Announcements

1) New homework up later today

2) Applications Monday
   - electrical circuits

Recall A norm on $\mathbb{R}^n$

is a function $\|\cdot\| : \mathbb{R}^n \to [0,\infty)$

such that for all real numbers

c and all x, y in $\mathbb{R}^n$

1) $\|x\| = 0$ only when $x = \vec{0}$

and $\|\vec{0}\| = 0$.

2) $\|cx\| = |c| \cdot \|x\|$

3) $\|x+y\| \leq \|x\| + \|y\|$

Showing that $\|\cdot\|_2$ is a norm on $\mathbb{R}^2$

Remember that if $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2}.$$

1) If $x = \vec{0}$, then $\|\vec{0}\|_2 = 0$.

$$\|\vec{0}\|_2 = \sqrt{0^2 + 0^2} = 0.$$

If $x \neq \vec{0}$, then either x_1 or x_2 is nonzero.

Then

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2}$$

$$> 0$$ since

one of x_1^2 or x_2^2 is positive

and the other is non-negative.

2) If c is any real number,

$$\|cx\|_2 = |c| \cdot \|x\|_2 .$$

$$cx = \begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix} .$$

$$\|cx\|_2 = \sqrt{(cx_1)^2 + (cx_2)^2}$$

$$= \sqrt{c^2 x_1^2 + c^2 x_2^2}$$

$$= \sqrt{c^2 (x_1^2 + x_2^2)}$$

$$= \sqrt{c^2} \sqrt{x_1^2 + x_2^2} = |c| \, \|x\|_2$$

3) If $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, then

$$\|x+y\|_2 \leq \|x\|_2 + \|y\|_2$$

$$x+y = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \end{bmatrix}$$

$$\|x+y\|_2 = \sqrt{(x_1+y_1)^2+(x_2+y_2)^2}$$

Is this less than or equal to

$$\|x\|_2+\|y\|_2 = \sqrt{x_1^2+x_2^2} + \sqrt{y_1^2+y_2^2} \; ?$$

Square $\|x+y\|_2$ and $\|x\|_2 + \|y\|_2$ to get

$$\|x+y\|_2^2 = (x_1+y_1)^2 + (x_2+y_2)^2$$

$$= \boxed{x_1^2 + 2x_1y_1 + y_1^2 + x_2^2 + 2x_2y_2 + y_2^2}$$

$$\left(\|x\|_2 + \|y\|_2\right)^2$$

$$= \boxed{\|x\|_2^2 + 2\|x\|_2\|y\|_2 + \|y\|_2^2}$$

We will show

$$\|x+y\|^2 \le \left(\|x\|_2 + \|y\|_2\right)^2$$

$$\|x+y\|_2^2$$

$$= x_1^2 + 2x_1y_1 + y_1^2 + x_2^2 + 2x_2y_2 + y_2^2$$

$$= (x_1^2 + x_2^2) + 2x_1y_1 + 2x_2y_2 + (y_1^2 + y_2^2)$$

$$= \|x\|_2^2 + 2(x_1y_1 + x_2y_2) + \|y\|_2^2$$

Is this less than or equal to

$$(\|x\|_2 + \|y\|_2)^2 = \|x\|_2^2 + 2\|x\|_2\|y\|_2 + \|y\|_2^2 ?$$

By subtracting off the common term $\|x\|_2^2 + \|y\|_2^2$, we reduce to showing that

$$\cancel{2}(x_1y_1 + x_2y_2) \le \cancel{2}\|x\|_2\|y\|_2$$

So is it true that

$$x_1y_1 + x_2y_2 \le \|x\|_2\|y\|_2 \ ?$$

Observe that $x_1y_1 + x_2y_2$

$$= x \cdot y$$

Remember that

$$x \cdot y = \|x\|_2 \|y\|_2 \cos\theta$$

$$\leq \|x\|_2 \|y\|_2 \cdot 1$$

Since $\cos\theta \leq 1$ for all values of θ. So we have shown

$$x_1 y_1 + x_2 y_2 = x \cdot y \leq \|x\|_2 \|y\|_2,$$

which will then get us

$$\|x+y\|_2 \leq \|x\|_2 + \|y\|_2.$$

Extending Linear Independence

We have defined what it means for one vector v to be linearly independent of $\{v_1, v_2, \dots, v_n\}$. We want to extend the notion of linear independence to sets of vectors.

Linear Independence for sets of vectors

$\{v_1, v_2, \ldots, v_n\}$ is linearly independent if none of the vectors are in the span of the others.

Quick check: This is the same as whenever

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = \vec{0}$$

for constants $c_1, c_2, \ldots, c_n$, then

$$c_1 = c_2 = \cdots = c_n = 0.$$

Matrix Interpretation

Row reduce the matrix

$$\begin{bmatrix} v_1 & v_2 & \cdots & v_n & \vec{0} \end{bmatrix}.$$

If we end up with

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & 0 & 1 & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & & & & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}$$

then the vectors are linearly independent. If not, they're not!

Example 1: Let

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 3 \\ 4 \\ 8 \\ 6 \end{bmatrix},$$

$$v_3 = \begin{bmatrix} 16 \\ 13 \\ -1 \\ -3 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 21 \\ -8 \\ 0 \\ 6 \end{bmatrix},$$

$$v_5 = \begin{bmatrix} 100 \\ 2 \\ -1000 \\ 8 \end{bmatrix}.$$

Is the set $\{v_1, v_2, v_3, v_4, v_5\}$ linearly independent?

Make the matrix

$$\begin{bmatrix} 1 & 3 & 16 & 21 & 100 & 0 \\ -1 & 4 & 13 & -8 & 2 & 0 \\ 0 & 8 & -1 & 0 & -1000 & 0 \\ 1 & 6 & -3 & 6 & 8 & 0 \end{bmatrix}$$

rref

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{3469}{2} & 0 \\ 0 & 1 & 0 & 0 & -\frac{911}{8} & 0 \\ 0 & 0 & 1 & 0 & 89 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1035}{8} & 0 \end{bmatrix}$$

So not linearly independent!

Orthogonality

Two vectors v and w in $\mathbb{R}^n$ are said to be **orthogonal** if

$$\boxed{v \cdot w = 0}$$

(angle between v and w is 90°)

A set of vectors $\{v_1, v_2, \ldots, v_m\}$ is orthogonal if any two vectors in the set are orthogonal.

Example 2:

$$v = \begin{bmatrix} -1 \\ 3 \\ 6 \end{bmatrix} \quad w = \begin{bmatrix} 6 \\ 0 \\ 1 \end{bmatrix}$$

are orthogonal since

$$v \cdot w = -6 + 0 + 6 = 0$$